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Introduction

We will consider some two dimensional flows of an incompressible heterogeneous ideal fluid of finite depth with a free surface. Let the coordinate axes be chosen so that when the fluid is at rest it occupies the strip $-h<y<0$, $-\infty< x< \infty$, of the x-y plane; the line $y = 0$ corresponds to the undisturbed free surface, and $y = -h$ to the rigid bottom. The density, ρ, of the fluid when at rest is assumed to depend on y only, and in order for the equilibrium to be stable, $\rho = \rho(y)$ must be a monotonic non-increasing function of y. The fluid is then said to be stably stratified.

The study of heterogeneous fluids was begun by Stokes (1847) who considered infinitesimal gravity waves in a fluid consisting of two layers of different density. This was later generalized to any number of layers by Webb and by Greenhill (see [2] §§ 231, 233). For a fluid of n layers with a free surface, this problem leads to the following conclusions. There are n "critical" velocities, $0< U_{n-1} < U_{n-2}, \ldots, < U_0< \infty$; for any velocity between 0 and U_{n-1} there are n possible waves; between U_{n-1} and U_{n-2} there are n-1 possible waves; and so on, until for $U > U_0$ no waves are possible. The shortest wave for any given speed is similar to the surface wave in a homogeneous medium. All the other waves are internal waves, having one or more nodes below the free surface. As the velocity approaches one of the critical speeds from below, the length of the longest wave increases monotonically to ∞. Some interest in the problem derives from the fact that under appropriate circumstances internal gravity waves can be very large and their effect quite noticeable, as, for example, in the case of "dead water" resistance (see [2], §§ 231, 249).

In contrast to the multilayered case, the problem of gravity waves for a fluid with a continuous density distribution has not been treated extensively. Although the equations were derived by Love in 1891 [4] only a few special cases have been investigated (see [2], § 235, [3], [6], and references quoted there). Current interest in the problem is in part due to the possible bearing on atmospheric motions.

This report is concerned with gravity waves of small amplitude in a fluid with a continuous density distribution. For the most part it will be assumed that $\rho(y)$ has a piecewise continuous derivative, $\rho'(y)$, and that $\rho'(y) < 0$, but is otherwise arbitrary. Section I deals with some of the general properties of progressive waves in the medium. This leads to a boundary value problem for an ordinary differential equation with two parameters, one of which also enters into the boundary condition at the free surface. As in the multilayer case, one concludes that there are a finite number of possible waves for any given velocity. However, the number is not limited, since there are an infinite number of critical speeds. Again, the shortest wave behaves in a different way from all the other waves. It is proved that there cannot exist waves with a speed greater than $(gh)^{1/2}$, the critical speed for a homogeneous medium. An example is given of a fluid of infinite depth which also exhibits an infinite number of critical speeds.

In Section II we examine the steady state problem for a surface pressure moving with a constant speed. An integral representation for the perturbed stream function is derived, and the behaviour of the flow far away from the pressure disturbance is investigated. A formula for the wave resistance is given, and from it the conditions for the vanishing of the resistance are deduced.

The uniqueness question for the steady problem suggests the study of the time dependent solution due to a moving surface pressure (see [7], pp. 210-218). The solution for general $\rho(y)$ is given in Section III. However, the proof that this solution approaches the steady solution is done only for the special case where $\rho = e^{-\beta y}$.

Section I. Waves with a free surface

A. The equations for the motion of a two dimensional incompressible fluid are:

(1) $$\rho(u_t + uu_x + vu_y) + p_x = 0$$

(2) $$\rho(v_t + uv_x + vv_y) + p_y + g\rho = 0$$

(3) $$\rho_t + (\rho u)_x + (\rho v)_y = 0$$

(4) $$\frac{d\rho}{dt} = \rho_t + u\rho_x + v\rho_y = 0$$

Here u and v are the velocity components in the x and y directions, p is the pressure, and ρ - the density. Equations (1) and (2) are the Euler equations of motion, (3) is the equation of mass conservation, and (4) is the incompressibility condition. In view

of (4), (3) can be replaced by the more convenient relation

(5) $$u_x + v_y = 0 \ .$$

The kinematic free surface condition is

(6) $$\frac{d\eta}{dt} = \eta_t + u\eta_x = v \qquad \text{at} \qquad y = \eta(x,t) \ .$$

If a surface pressure, $P(x,t)$, is applied, the dynamic condition is

(7) $$p(x,\eta(x,t),t) = P(x,t) \ .$$

At the bottom we have

(8) $$v = 0 \qquad \text{at} \qquad y = -h \ .$$

If $P \equiv 0$ and the fluid is in a state of rest, the solution is

$$u \equiv v \equiv 0, \qquad \eta \equiv 0, \qquad p = g \int_y^{\eta} \rho(s)ds$$

where $\rho = \rho(y)$ is continuous and monotonic decreasing with y. The equations for a small disturbance from the state of rest, brought about by the application of a small surface pressure, can be obtained formally by the standard perturbation procedure. Since equation (5) for the perturbed quantities does not change in form, we can introduce a stream function, ψ, for the perturbed velocities. We will designate these velocities by u and v again, since no other velocities will be referred to from this point on. Thus, take $\psi(x,y,t)$ such that

$$\psi_y = u, \qquad \psi_x = -v$$

We can eliminate the perturbed pressure and density from the equations, and finally obtain

(9) $$[(\rho\psi_x)_x + (\rho\psi_y)_y]_{tt} - g\rho'\psi_{xx} = 0$$

where $\rho = \rho(y)$ is the originally prescribed density distribution. This is Love's equation (see [4]). Similarly, we can derive the boundary condition

(10) $$\psi_{ytt} - g\psi_{xx} = -\rho_o^{-1} P_{xt}$$

which is to be satisfied at $y = 0$. Here $\rho_o = \rho(0)$. The boundary condition at $y = -h$ becomes

(11) $$\psi = \text{const} = 0 .$$

The boundary condition (10) is correct only if P and ψ have the required derivatives, which is not the usual situation of interest in hydrodynamics. It will be replaced when necessary later.

B. We will now investigate the possible existence of waves with velocity U, i.e. of solutions, $\psi(x,y,t)$, which have the form

$$\psi(x,y,t) = e^{ik(x\pm Ut)}Y(y)$$

With k real. Such waves exist if one can find non-trivial $Y(y)$ satisfying the boundary value problem

(12) $$(\rho Y')' - k^2\rho Y - \mu\rho' Y = 0, \quad -h < y < 0$$

(13) $$Y' - \mu Y = 0 \qquad \text{at } Y = 0$$

(14) $$Y = 0 \qquad \text{at } Y = -h$$

where $\mu = gU^{-2}$. This problem can be regarded in two ways: either μ is held fixed and $\lambda = -k^2$ is the eigenvalue parameter, or k is held fixed and μ is the eigenvalue parameter.

First, suppose μ is held fixed. We look for possible values of k for which non-trivial solutions to (12) - (14) exist, i.e. for possible waves having a fixed velocity. This is a regular Sturm-Liouville problem, and it is well known that there exists a discrete spectrum of real (simple) eigenvalues, $-k_n^2$, and associated eigenfunctions, $\phi_n(y)$, which are orthogonal in the sense that

$$\int_{-h}^{0} \rho(y)\phi_n(y)\phi_m(y)dy = 0 \quad \text{if } n \neq m.$$

The eigenvalues and eigenfunctions can also be characterised by means of a variational problem. Let

(15) $$(Y,L_1Y) = \int_{-h}^{0} (\rho Y'^2 + \mu\rho' Y^2)dy - \mu\rho_o Y^2(0)$$

and

(16) $$(Y,Y) = \|Y\|_1^2 = \int_{-h}^{0} \rho Y^2 dy$$

If Y(y) satisfies the boundary condition Y(-h) = 0, integration by parts yields

(17) $$(Y,L_1Y) = \int_{-h}^{0} \rho(Y'^2 - 2\mu YY')dy$$

$$= \int_{-h}^{0} \rho[(Y' - \mu Y)^2 - \mu^2 Y^2]dy$$

and therefore,

(18) $$(Y,L_1Y) \geq -\mu^2 \|Y\|_1^2$$

The eigenvalues, $-k_n^2$, and the eigenfunctions, ϕ_m, can be obtained by minimizing (Y,L_1Y) with $\|Y\|_1^2 = 1$, $Y(-h) = 0$, and with Y satisfying the orthogonality conditions

$$(Y,\phi_o) = \ldots = (Y,\phi_{n-1}) = 0 .$$

Consequently, there exists a sequence of k_n^2 with

$$\mu^2 \geq k_0^2 > k_1^2 > k_2^2 > \dots > k_n^2 > \dots$$

and $k_n^2 \longrightarrow \infty$ as $n \longrightarrow \infty$. Only a finite number of the k_n^2 can be positive, and therefore, there can only be a finite number of bounded waves (if any) for each velocity. The eigenfunction ϕ_0, which corresponds to the shortest wave, has no zeros in $-h < y < 0$; it is similar to the usual surface wave in a homogeneous fluid. The eigenfunction ϕ_n has n zeros, and at least n internal maxima and minima, and the corresponding wave can be called an internal wave.

It is useful to examine the maximum-minimum property of the eigenvalues. Every eigenvalue, $-k_n^2$, can be found by taking the least upper bound of the minima of (Y,L_1Y) with $\|Y\|_1^2 = 1$, $Y(-h) = 0$, and n other independent linear constraints ([1]),ch.6). Then it is clear from (15) that if $\rho' < 0$, k_n^2 is a non-decreasing function of μ. This suggests the possibility that for some value of μ one of the k_n^2 can vanish; such values of μ would correspond to critical speeds. This will be clearer when we examine the problem with fixed k and varying μ.

Remark. Using the orthogonality and completeness of the $\phi_n(y)$ makes it easy to write down a proof, similar to Weinstein's for a homogeneous medium [9], that $e^{ik_n(x \pm Ut)}\phi_n(y)$ (k_n real) are the only bounded waves possible among ψ with $\int_{-h}^{0} \rho(y)\psi(x,y,t)dy < \infty$.

One need only expand ψ in a series of the ϕ_n,

$$\psi(\xi,y) = \sum c_n(\xi)\phi_n(y)$$

where $\xi = x \pm Ut$, and the c_n will be found to satisfy

$$c_n'' + k_n^2\, c_n = 0 \; .$$

C. Let k be held fixed, and let μ be the eigenvalue parameter. The problem

$$L_2 Y = -(\rho Y')' + k^2 \rho Y = -\mu\rho' Y$$

with $\qquad Y' = \mu Y$ at $y = 0$ and $Y(-h) = 0$

is not a standard Sturm-Liouville problem since μ occurs in the boundary conditions. However, this complication introduces no difficulties. Let

(19) $$(Y, L_2 Y) = \int_{-h}^{0} \rho(Y'^2 + k^2 Y^2)dy$$

and

(20) $$(Y,Y) = \|Y\|_2^2 = -\int_{-h}^{0} \rho' Y^2 dy + \rho_o Y^2(0) \; .$$

(Here we require $\rho' < 0$.) It is easy to see that for Y satisfying $Y(-h) = 0$, $(Y, L_2 Y)$ is positive definite with respect to $\|Y\|_2^2$. Integration by parts gives

$$\|Y\|_2^2 = 2\int_{-h}^{0} \rho Y Y' dy$$

and then

(21) $$\|Y\|_2^{-2} \int_{-h}^{0} \rho(Y'^2 + k^2Y^2)dy \geq \int_{-h}^{0} \rho Y'^2 dy \left[2\int_{-h}^{0} \rho Y Y' dy \right]^{-1}$$

$$\geq \frac{1}{2}\int_{-h}^{0} \rho Y'^2 dy \left[\int_{-h}^{0} \rho Y^2 dy \int_{-h}^{0} \rho Y'^2 dy \right]^{-1/2}$$

$$\geq \frac{1}{2}\left[\int_{-h}^{0} \rho Y'^2 dy\right]^{1/2}\left[\int_{-h}^{0} \rho Y^2 dy\right]^{-1/2}$$

$$> \frac{1}{2}\left[\frac{\rho(o)}{\rho(-h)}\right]^{1/2}\left[\int_{-h}^{0} Y'^2 dy\right]^{1/2}\left[\int_{-h}^{0} Y^2 dy\right]^{-1/2}$$

$$> \left[\frac{\rho(o)}{\rho(-h)}\right]^{1/2} \frac{\pi}{2h}$$

One can conclude (see for example [5]) that there exists a discrete spectrum of eigenvalues, $\mu_n = \mu_n(k^2)$, with

$$0 < \mu_o < \mu_1 < \mu_2 < \ldots < \mu_n < \ldots$$

and $\mu_n \to \infty$ as $n \to \infty$. The corresponding sequence of eigenfunctions, $Y_n = Y_n(y;k)$, are complete and orthogonal with respect to the inner product (20), i.e.

(22). $$(Y_n, Y_m) = -\int_{-h}^{0} \rho' Y_n Y_m dy + \rho_o Y_n(0) Y_m(0) = 0 \quad \text{if } n \neq m.$$

An arbitrary function, $f(y)$, defined in $-h < y < 0$, with $\|f\|_2 < \infty$, can be expanded in a series with respect to the eigenfunctions (for any real k):

(23). $$f(y) = \sum_{o}^{\infty} a_n Y_n(y)$$

where

(24). $$a_n = \|Y_n\|_2^{-2}\left[\int_{-h}^{0} \rho' Y_n f dy + \rho_o Y_n(0) f(0)\right]$$

All these conclusions hold for any fixed real k, and in particular for k = 0. This proves that there is an infinite number of critical speeds, $U_n = [g/U_n(0)]^{1/2}$, and $U_n \to 0$ as $n \to \infty$.

It is interesting to observe some of the features of μ_n as a function of k^2 (which characterises the dispersiveness of the

fluid) without making any further restrictions on the density distribution. The maximum-minimum property of μ_n, in conjunction with (19), shows that $\mu_n(k^2)$ is monotonic non-decreasing. We can also make use of the fact that μ_n is an analytic function of k^2. A formal perturbation then yields

$$(25) \qquad \frac{d\mu_n}{dk^2} = \frac{\int_{-h}^{0} \rho Y_n^2 dy}{\|Y_n\|_2^2} = \frac{\mu_n}{k^2} - \frac{\int_{-h}^{0} \rho Y_n'^2 dy}{k^2\|Y_n\|_2^2}$$

which shows that $\mu_n(k^2)$ is strictly monotonic and that $0 < k^{-2}\mu_n < c$ $(c > 0)$. Furthermore, since

$$k^2 \frac{d}{dk^2}\left(\frac{\mu_n}{k^2}\right) = \frac{d\mu_n}{dk^2} - \frac{\mu_n}{k^2}$$

we obtain

$$(26) \qquad \frac{d}{dk^2}\left(\frac{\mu_n}{k^2}\right) < 0$$

Consequently, $k^{-2}\mu_n$ approaches a limiting value as $k^2 \longrightarrow \infty$. Formula (25) also states that at every point of the $\mu_n(k^2)$ curve the slope of the tangent to the curve is less than the slope of the line from the origin to the point.

If $\nu_n = \nu_n(k) = g\mu_n^{-1/2}$, the progressive waves can be written in the form

$$\psi_n = e^{ik(x \pm \nu_n t)} Y_n(y;k)$$

then the group velocity of the wave is

$$(27) \qquad V_{gn} = (k\nu_n)' = k\nu_n' + \nu_n$$

where ' denotes differentiation with respect to k; (26) implies that $V_{gn} > 0$. We can define $\nu_n(k)$ to be an even function of k, monotonically decreasing from $U_n = \nu_n(0)$ to zero as $|k| \longrightarrow \infty$. Consequently $k\nu_n' \leq 0$, and it follows from (27) that the group velocity is always less than the phase velocity.

All the above properties of μ_n and ν_n hold for all n. It might be expected that in some ways the surface wave will differ from the internal waves. One such point of difference is the asymptotic behaviour (as $k^2 \longrightarrow \infty$). Since

$$(28) \qquad \frac{\rho_o \int_{-h}^{0} (Y'^2 + k^2 Y)dy}{r_1 \int_{-h}^{0} Y^2 dy + \rho_o Y^2(0)} \leq \frac{(Y, L_2 Y)}{\|Y\|_2^2} \leq \frac{\rho(-h) \int_{-h}^{0} (Y'^2 + k^2 Y)dy}{r_2 \int_{-h}^{0} Y^2 dy + \rho_o Y^2(0)}$$

where $r_1 = \max(-\rho')$ and $r_2 = \min(-\rho')$, we can obtain bounds for μ_n from the eigenvalues of two auxiliary problems. If α_n are the eigenvalues of the problem

$$(29) \qquad Y'' - k^2 Y + \alpha \frac{r_1}{\rho_o} Y = 0$$

$$(29') \qquad Y(-h) = 0, \quad Y' - \alpha Y = 0 \quad \text{at } y = 0$$

and β_n the eigenvalues of

$$(30) \qquad Y'' - k^2 Y + \beta \frac{r_2}{\rho(-h)} Y = 0$$

$$(30') \qquad Y(-h) = 0, \quad Y' - \beta \frac{\rho_o}{\rho(-h)} Y = 0 \quad \text{at } Y = 0$$

then $\alpha_n \leq \mu_n \leq \beta_n$. It is easy to obtain the asymptotic behaviour of α_n and β_n, and we finally get

$$(31) \qquad 0 < A_o \leq \frac{\mu_o}{|k|} \leq B_o$$

for large k^2, and

(31') $$0 < A_n \leq \frac{\mu_n}{k^2 + (\frac{n\pi}{h})^2} \leq B_n, \quad n = 1, 2, 3, \ldots$$

for large $k^2 + (\frac{n\pi}{h})^2$. Thus ν_o is of the order of the square root of the wavelength, while ν_n is of the order of the wavelength, as the wavelength goes to zero.

D. It was shown in the previous section that for any admissible ρ (i.e. $\rho > 0$, $\rho' < 0$) all waves travel with a speed less than U_o, the highest critical speed. It is natural to ask if there is a wave of greatest speed, and for what density distribution can it occur. This is equivalent to finding μ^*,

$$\mu^* = \text{glb } \mu_o(0)$$

among all admissible ρ. We will show that $\mu^* = h^{-1}$, i.e. that $U_o < \sqrt{gh}$, which is the critical speed in a homogeneous medium. This can be easily done by employing Jacobi's multiplicative variation.

It is sufficient to prove that

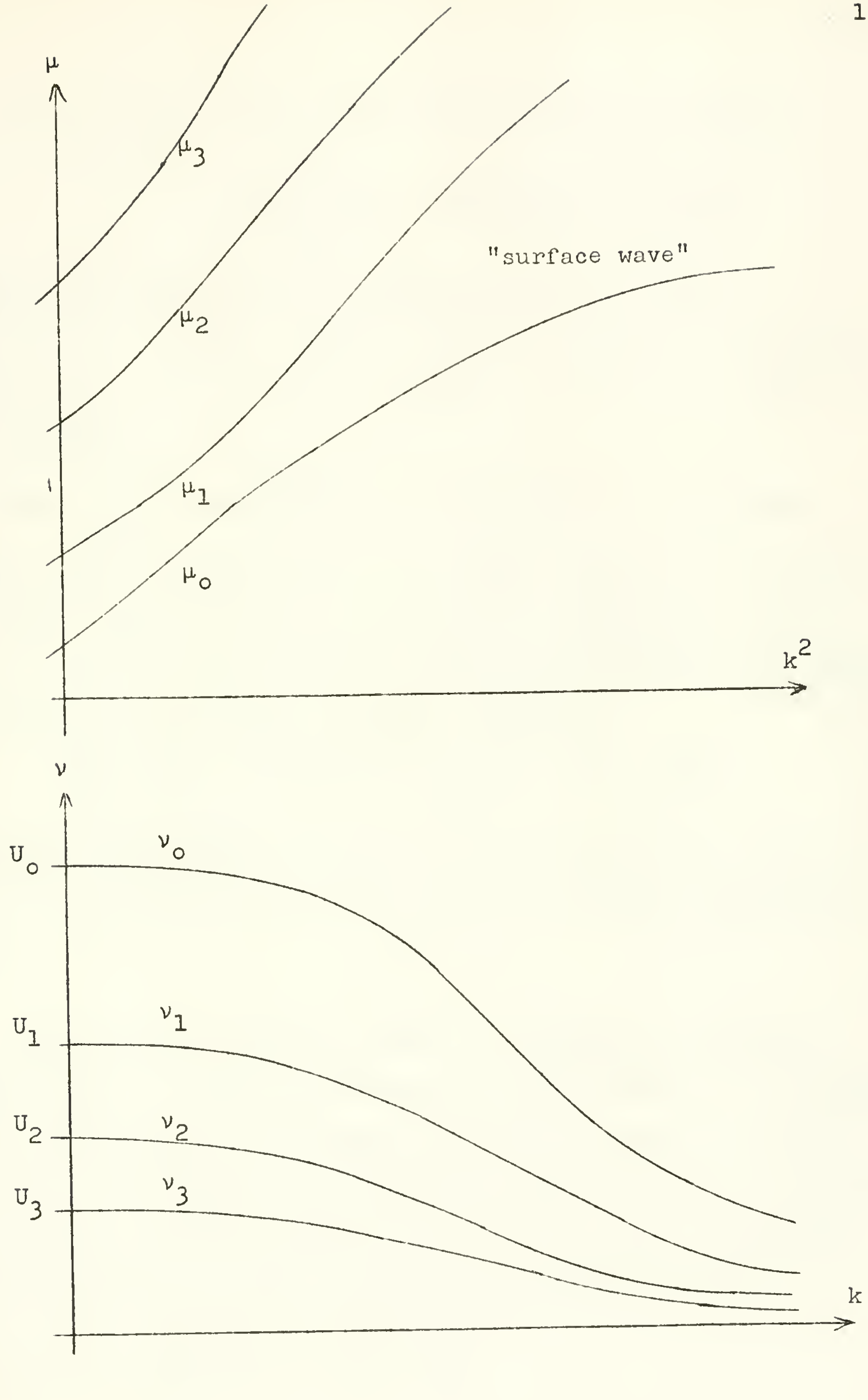
(32) $$Q(Y) = \int_{-h}^{0} \rho Y'^2 dy + h^{-1}\left[\int_{-h}^{0} \rho' Y^2 dy - \rho_o Y^2(0)\right] > 0$$

for all admissible Y and ρ (Y is admissible if it has a piecewise continuous derivative and if $Y(-h) = 0$). Let the function $Z(y)$ be defined by

(33) $$Z(y) = h(h + y)^{-1}\, Y(y)$$

and $$Z(-h) = hY'(-h)$$

Then $Z(y)$ is continuous and has a piecewise continuous derivative. Substituting (33) in (32) gives



μ
μ_3
μ_2
μ_1
μ_0
"surface wave"
k^2
ν
U_0
U_1
U_2
U_3
ν_0
ν_1
ν_2
ν_3
k

$$Q(Y) = \int_{-h}^{0} \rho\left[(1 + y/h)^2 Z'^2 + h^{-2}Z^2 + 2h^{-1}(1 + y/h)ZZ'\right]dy$$
$$+ h^{-1}\int_{-h}^{0} \rho'(1 + y/h)^2 Z^2 dy - h^{-1}\rho_0 Z^2(0).$$

Integrating by parts the term containing ZZ', we obtain

$$\int_{-h}^{0} 2ZZ'\rho(1 + y/h)dy = \rho_0 Z^2(0) - \int_{-h}^{0} \rho'(1 + y/h)Z^2 dy - h^{-1}\int_{-h}^{0} \rho Z^2 dy$$

and therefore,

$$(34)\quad Q(Y) = \int_{-h}^{0} \rho(1 + y/h)^2 Z'^2 dy + h^{-1}\int_{-h}^{0} (y/h)\,(1 + y/h)\rho' Z^2 dy$$

Clearly, $Q(Y) \geq 0$ since $\rho > 0$, $(1 + y/h) > 0$ and $\rho' y > 0$, and Q can vanish only if $Z' \equiv \rho' \equiv 0$. Thus, $Q = 0$ only if $\rho = \text{const}$ and $Y = \text{const}\,(1 + y/h)$, which is the eigenfunction for the problem (12)-(14) with $k = 0$ and $\rho = \text{const}$. Therefore,

$$\operatorname*{glb}_{\rho}\, \mu_0(0) = \mu^* = h^{-1} \quad \text{or} \quad \operatorname*{lub}_{\rho}\, U_0 = U^* = \sqrt{gh}\ .$$

It might be noted that as $h \longrightarrow \infty$, $\mu^* \longrightarrow 0$ and $U^* \longrightarrow \infty$, i.e. there is no critical speed for an ocean of infinite depth and $\rho =$ const.

E. For an ocean of infinite depth the situation is much more complicated since the Sturm-Liouville problem (12)-(14) is singular. We will not discuss this problem here, except to give an example with continuous ρ which has an infinite number of critical speeds. This may have some importance since it suggests the possibility of the existence of solitary waves in an infinite domain.

$$\text{Let } \rho(y) = \begin{cases} \rho_0 e^{-\beta y}, & -h < y < 0 \\ \rho_0 e^{-\beta h}, & -\infty < y < -h \end{cases}$$

Below $y = -h$, $Y = Y_1$ satisfies

$$Y_1'' - k^2 Y_1 = 0 \tag{34}$$

while for $-h < y < 0$, it satisfies

$$Y'' - \beta Y' + \mu\beta Y - k^2 Y = 0 \ . \tag{35}$$

Therefore,

$$Y_1 = e^{|k|y}$$

(in fact $\quad Y_1' - |k| Y_1 = 0 \quad \text{for} \quad y \leq -h)$.

In $-h < y < 0$,

$$Y = e^{1/2\ \beta y}[A \cosh \alpha(y + h) + \beta \sinh \alpha(y + h)]$$

where

$$\alpha^2 = \frac{1}{4}\beta^2 - \mu\beta + k^2 \ .$$

Since Y and Y' must be continuous at $y = -h$, we can evaluate A and B. The free boundary condition then yields the following equation to determine μ:

$$\frac{\tanh \alpha h}{\alpha h} = \frac{\mu - |k|}{|kh|(|k| + \frac{1}{2}\beta - \mu) - \frac{1}{2}\mu\beta h} \ . \tag{36}$$

The critical speeds are obtained from this equation with $k = 0$:

$$\tanh \alpha h = -2\beta^{-1}\alpha \tag{37}$$

and $\quad \alpha^2 = \frac{1}{4}\beta^2 - \mu\beta$.

There are an infinite number of solutions, $\mu_n(0)$, with all the $\mu_n(0) < \frac{1}{4}\beta$.

Section II Travelling surface pressure; steady solution.

A. In this section we will consider the disturbance created by a moving pressure distribution applied at the surface. Suppose a steady pressure is applied at t = 0, and moves with a constant velocity, U, in the direction of the negative x-axis, i.e.

$$P = P(x + Ut) = P(\xi), \quad \xi = x + Ut. \tag{38}$$

We assume P is continuous, and vanishes outside of the finite interval, $|\xi| \leq \ell$. The derivative, $P'(\xi)$, need not be continuous; however, we will require $P(\xi)$ to behave no worse than $(\ell^2-\xi^2)^\varepsilon$, $0 < \varepsilon < 1$, as ξ approaches either end point from the interior.

It is natural to expect that when viewed from a coordinate system moving with the pressure, the disturbance will tend to a steady state. Consequently, we look for a solution of the form $\psi = \Psi(\xi,y)$. Introducing this into (9) and (10) of the preceding section yields

$$U^2[\rho\Delta\Psi + \rho'\Psi_y]_{\xi\xi} - g\rho'\Psi_{\xi\xi} = 0 \tag{39}$$

$$[U^2\Psi_y - g\Psi]_{\xi\xi} = - U\rho_o^{-1}P_{\xi\xi} \qquad \text{at } y = 0. \tag{40}$$

The BC (40) is not suitable, since even P_ξ may become infinite at $\xi = \pm\ell$. However, we can write it in the form

$$[\Psi_y - \mu\Psi + \frac{1}{\rho_o U} P]_{\xi\xi} = 0$$

and try to satisfy the condition that the quantity inside the brackets vanish (If we go back to the original equations and look

for a solution which depends on ξ and y only, the superfluous differentiations will also not appear). We obtain the problem

(41) $$(\rho\Psi_\xi)_\xi + (\rho\Psi_y)_y - \mu\rho'\Psi = 0, \qquad \mu = g/U^2$$

(42) $$\Psi_y - \mu\Psi = -\frac{P}{\rho_0 U} \qquad \text{at } y = 0$$

(43) $$\Psi = 0 \qquad \text{at } y = -h.$$

As is common in situations of this type, the problem defined by (41)-(43) has in general more than one solution. It is clear that for speeds below the highest critical speed (for the given ρ), all the possible waves are solutions to the homogeneous problem ($P = 0$), and can be added to Ψ. To insure uniqueness one can introduce an artificial "viscosity" and find the limit solution as this parameter vanishes (Rayleigh), or one can impose conditions on the behaviour of the disturbance at ∞. We will require that Ψ vanish far ahead of the disturbance ($\xi \longrightarrow -\infty$), and that it be bounded as $\xi \longrightarrow +\infty$. Then, uniqueness among the class of functions with $\int_{-h}^{0} \rho(y)\Psi^2(\xi,y)dy < \infty$, for all ξ, can be easily proved by using Weinstein's procedure (see remark in IB). A more satisfactory justification can be made by considering the time dependent solution, and showing that it approaches the steady solution (see III).

B. The solution to the problem (41)-(43) with the condition that Ψ vanish far "upstream" can be obtained by a superposition of solutions to (41) which are harmonic in ξ and satisfy the boundary condition at the rigid bottom. We know from previous results that $e^{ik\xi}Y(y;k)$ is a solution to (41) if it satisfies

(44) $$(\rho Y')' - k^2\rho Y - \mu\rho' Y = 0.$$

Choose Y so that

(45) $$Y = 0 \quad \text{and } Y' = 1 \quad \text{at } y = -h.$$

Then Y is uniquely determined. It should be noted for later use that Y(y;k) is an analytic entire function of k for each y. Now consider a superposition of such solutions for all possible k:

$$\Psi(\xi,y) = \int_L A(k)e^{ik\xi}Y(y;k)dk$$

where the path of integration in the complex k-plane will be fixed later. To satisfy the boundary condition at y = 0 formally

$$\Psi_y(\xi,0) - \mu\Psi(\xi,0) = \int_L A(k)[Y' - \mu Y]_o\, e^{ik\xi}dk = -\frac{1}{\rho_o U}P(\xi)$$

where the subscript o doubles the value at y = 0. If the path is such that the Fourier Integral Theorem can be applied (formally), we can evaluate A(k), obtaining

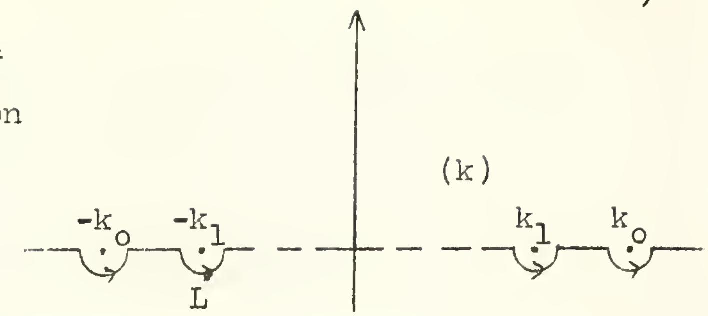
(46) $$\Psi(\xi,y) = -(2\pi\rho_o U)^{-1}\int_L \frac{F(k)\ Y(y;k)}{[Y' - \mu Y]_o} e^{ik\xi}dk$$

where

$$F(k) = \int_{-\ell}^{\ell} P(\xi)e^{-ik\xi}d\xi\ .$$

The denominator of the integrand has an infinite number of zeros on the imaginary axis, and if $\mu > \mu_o(0)$, a finite number on the real axis. Suppose $\mu > \mu_o(0)$ but is not one of the critical speeds, and let $k = \pm k_n$, n = 0, 1, 2, ..., N be the zeros on the real axis ($k_n \neq 0$). Then for L we pick the path which coincides with the real k-axis except near $k = \pm k_n$, where it consists of semicircles in the lower half plane. It will be

proved now that $\tilde{\psi}$, as given by (46) is indeed a solution to (41)-(43) and satisfies the required condition as $\xi \longrightarrow -\infty$.



C. To prove that (46) is a solution we need the behaviour of the integrand for large $|k|$, k real. The asymptotic behaviour of Y is easy to obtain if we change variables. Let $V = \rho^{-1/2}Y$. Then (44) and (45) become

$$(47) \qquad V'' - [\mu \frac{\rho'}{\rho} + \frac{1}{2}\frac{\rho''}{\rho} - \frac{1}{4}(\frac{\rho}{\rho})^2 + k^2]V = 0$$

$$(48) \qquad V(-h) = 0, \; V'(-h) = \sqrt{\rho(-h)} = 1 \text{ if we normalize } \rho.$$

Then it is easy to see that

$$(49) \qquad Y = \rho^{-1/2}k^{-1} \sinh k(y + h)[1 + O(k^{-1})]$$

$$Y' = \rho^{-1/2} \cosh k(y + h)[1 + O(k^{-1})]$$

and

$$Y' - \mu Y = \rho^{-1/2} \cosh k(y + h)[1 + O(k^{-1})]$$

uniformly for $-h \leq y \leq 0$ and large $|k|$. Therefore,

$$(50) \qquad \frac{Y(y;k)}{[Y' - \mu Y]_o} = \sqrt{\frac{\rho_o}{\rho}} \frac{e^{|k|y}}{|k|} [1 + O(k^{-1})].$$

From the assumptions made about $P(\xi)$, we know that $F(k) = O(k^{-1-\epsilon})$, $0 < \epsilon < 1$. Consequently, the integrand is exponentially bounded for large $|k|$ and $y < 0$, and therefore, the integral in (46) is absolutely convergent. Furthermore, it can be differentiated any number of times with respect to ξ and y under the integral sign, and this proves that (46) is a solution to the differential equation (41).

To show that (46) satisfies the boundary condition (42), form $\Psi_y - \mu\Psi$ for $y < 0$, and let $y \longrightarrow 0$.

$$\Psi_y - \mu\Psi = -\frac{1}{2\pi\rho_o U}\int_L F(k)e^{ik\xi}\,\frac{Y' - \mu Y}{[Y' - \mu Y]_o}\,dk$$

Since $F(k) = O(k^{-1-\varepsilon})$ and $\frac{Y' - \mu Y}{[Y' \mu Y]_o}$ certainly stays bounded as $y \longrightarrow 0$, the integral on the right hand side is absolutely convergent for $y = 0$, and therefore we can go to the limit under the integral, obtaining

$$\lim_{y=0}\,[\Psi_y - \mu\Psi] = -\frac{1}{2\pi\rho_o U}\int_L F(k)e^{ik\xi}\,dk.$$

$F(k)$ is analytic, of course, and we can deform L to the real axis:

$$(51)\qquad [\Psi_y - \mu\Psi]_o = -\frac{1}{2\pi\rho_o U}\int_{-\infty}^{\infty} F(k)e^{ik\xi}\,dk$$
$$= -\frac{1}{\rho_o U}\,P(\xi)$$

which proves that (42) is satisfied. Obviously (43) is also satisfied.

Before going on to the asymptotic behaviour of $\Psi(\xi,y)$, we note that for $y < 0$ it is certainly permissible to substitute the expression for $F(k)$ into (46) and then to change the order of integration, which results in

$$(52)\qquad \Psi(\xi,y) = -\frac{1}{\rho_o U}\int_{-\ell}^{\ell} P(s)\,K(\xi,s;y)\,ds$$

where

$$(53)\qquad K(\xi,s;y) = \frac{1}{2\pi}\int_L \frac{Y(y;k)}{[Y'-\mu Y]_o}\,e^{ik(s-\xi)}\,dk$$

is the stream function due to a δ-function in the surface pressure.

In the neighborhood of $s = \xi$ and $y = 0$, k has a logarithmic singularity.

It will now be shown that Ψ vanishes as $\xi \to -\infty$. Let the integral in (46) be expressed as the sum of two integrals, one over a portion of the real k-axis and the other over the 2(N+1) semicircles. Since the integrand is $O(k^{-2-\varepsilon})$ for large $|k|$ on the real axis, the first contribution vanishes as $|\xi| \to \infty$ by the Riemann-Lebesgue Lemma. To evaluate the second contribution, consider one of the semicircles, say the one near $k = k_n$. The integrand can be written in the form $f(y;k)e^{i(k-k_n)\xi}$, with f bounded on the semicircle. On the semicircle, $k-k_n = \alpha(\cos\theta + i \sin\theta)$, $\alpha > 0$, $\pi \leq \theta \leq 2\pi$, and so we obtain an integral of the form

$$\int_{\pi}^{2\pi} f_1(y,\theta)e^{-\alpha \sin\theta \cdot \xi}\, d\theta$$

with f_1 bounded. As $\xi \to -\infty$ the integral vanishes like $|\xi|^{-1/2}$. Since there is a finite number of semicircles, the second contribution also vanishes.

D. To complete the proof that (46) is the steady state stream function, we now want to examine the principal effect of the surface disturbance: the waves created in the wake of the disturbance. The surface elevation, η is given by

$$\eta = -\frac{1}{U}\lim_{y=0} \Psi(\xi,y) .$$

Consequently we will examine Ψ as $\xi \to +\infty$. In order to do this we need the behaviour of $[Y' - \mu Y]_o$ in the complex k-plane near its zeros, $k = \pm k_n$, and this we proceed to do first.

Suppose $\rho(y)$ is an eigenfunction and $-k^2 < 0$ is the corresponding eigenvalue of

(54) $$L_1 Y = -(\rho Y')' + \mu\rho' Y = -k^2 \rho Y$$

(55) $$Y(-h) = [Y' - \mu Y]_o = 0 ,$$

i.e.

(56) $$L_1 \phi = -k^2 \rho\phi$$

(57) $$\phi(-h) = [\phi' - \mu\phi]_o = 0$$

and normalize ϕ to have $\phi'(-h) = 1$. Now let k^2 be a complex number in the neighborhood of k_*^2, and let $Y(y;k)$ be the solution of (54) which satisfies

$$Y(-h;k) = 0 , \quad Y'(-h;k) = 1 .$$

The differential equation (54) can be written in the form

(58) $$L_1 Y + k_*^2 \rho Y = -(k^2 - k_*^2)\rho Y .$$

Consequently, (54) can be replaced by the integral equation

(59) $$Y = \phi(y) - (k^2 - k_*^2) \int_{-h}^{y} [\phi(y)\chi(\eta) - \phi(\eta)\chi(y)]\rho(\eta) Y(\eta;k)\,d\eta$$

where $\chi(y)$ is the solution to

(60) $$L_1 \chi = - k_*^2 \rho \chi$$

(61) $$\chi(-h) = - \frac{1}{\rho(-h)} , \quad \chi'(-h) = 0 .$$

Then

(62) $$Y' - \mu Y = (\phi' - \mu\phi) - (k^2 - k_*^2) \int_{-h}^{y} \{ [\phi'(y) - \mu\phi(y)] \chi(\eta)$$
$$- [\chi'(y) - \mu \chi(y)] \phi(\eta) \} \rho(\eta) Y(\eta;k) d\eta .$$

Letting $y \to 0$, we obtain

(63) $$\lim_{y=0} [Y' - \mu Y] = [Y' - \mu Y]_o = [\phi - \mu\phi]_o$$
$$- (k^2 - k_*^2) \int_{-h}^{0} [\phi' - \mu\phi]_o \chi(\eta) - [\chi' - \mu \chi]_o \phi(\eta) P(\eta) Y(\eta;k) d\eta$$
$$= (k^2 - k_*^2) [\chi' - \mu \chi]_o \int_{-h}^{0} \rho(\eta)\phi(\eta) Y(\eta;k) d\eta$$

$[\chi' - \mu \chi]_o$ cannot vanish, and $Y(y;k) \to \phi(y)$ uniformly in y as $k^2 \to k_*^2$. Therefore,

(64) $$[Y' - \mu Y]_o = (k^2 - k_*^2) [\chi' - \mu \chi]_o \int_{-h}^{0} \rho(\eta)\phi^2(\eta) d\eta + O((k - k_*)^2)$$

since Y and Y' are analytic in k for every fixed y. This shows that $[Y' - \mu Y]_o$ has only simple zeros on the real k-axis (provided μ does not correspond to a critical speed). We can also evaluate $[\chi' - \mu \chi]_o$ since we know the values of the Wronskian, W:

$$W(y) = \phi \chi' - \chi \phi' = \frac{1}{\rho(y)} .$$

Therefore,

$$W(0) = [\chi' - \mu\chi]_0 \phi(0) = \frac{1}{\rho_0}$$

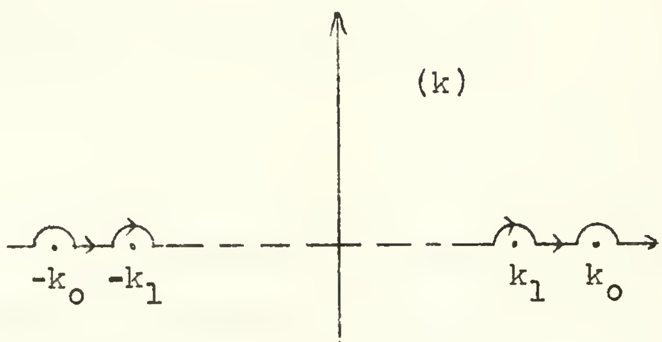
and this yields finally

$$(65) \qquad [Y' - \mu Y]_0 = \frac{(k^2 - k_*^2)}{\rho_0 \phi(0)} \int_{-h}^{0} \rho(y)\phi^2(y)dy + O((k - k_*)^2) \quad .$$

E. In order to evaluate $\lim\limits_{\xi \to +\infty} \Psi(\xi, y)$, we can deform the path L into L^+, which coincides with L on the real axis, but passes above the singularities. Clearly,

$$\int_L = \int_{L^+} + 2\pi \sum \text{residues at } k = k_n \; .$$

As $\xi \to +\infty$ the integrand is exponentially bounded on the semicircle of L^+, and consequently $\int_{L^+} \to 0$, leaving only the contribution at the singularities. Using (65) we can compute the residues, obtaining (as $\xi \to +\infty$),



$$(66) \qquad \Psi(\xi, y) \to \frac{1}{2iU} \sum_{\pm k_n} \frac{Y(0;k_n)Y(y;k_n)}{k_n \int_{-h}^{0} \rho Y^2(y;k_n)dy} F(k_n)e^{ik_n\xi}$$

where the summation extended over all the poles ($k = \pm k_n$, $\eta = 0,1,\ldots,N$). Since Y is an even function of k, we can combine the contributions at $+k_n$ and $-k_n$:

$$(67) \qquad \Psi(\xi,y) \rightarrow U^{-1} \sum \frac{Y(0;k_n)Y(y;k_n)}{k_n \int_{-h}^{0} \rho Y^2(y;k_n)dy} \int_{-\ell}^{\ell} P(s) \sin k_n(\xi-s)ds$$

where the summation is now over the positive k_n, or using the designation $\phi_n(y)$ for $Y(y;k_n)$ as in IB,

$$(68) \qquad \Psi(\xi,y) \rightarrow U^{-1} \sum \frac{\phi_n(0)\phi_n(y)}{k_n \|\phi_n\|^2} \int_{-\ell}^{\ell} P(s) \sin k_n(\xi-s)ds$$

and

$$(69) \qquad \eta(\xi) \rightarrow -\frac{1}{U^2} \sum \frac{\phi_n^2(0)}{k_n \|\phi_n\|^2} \int_{-\ell}^{\ell} P(s) \sin k_n(\xi-s)ds$$

as $\xi \rightarrow +\infty$.

We now have a picture of the disturbance. When the pressure, P, moves at a speed greater than the highest critical speed, U_0 (i.e. $\mu < \mu_0(0)$), there are no waves far ahead or far behind P. If $U_1 < U < U_0$, there is one wave; its length decreases as U decreases. Below U_1 a second wave appears; its length is greater than that of the first wave, and also decreases as U decreases. Below U_2 a third wave appears, and so on. Far away these waves behave like the progressive waves corresponding to the speed U. It should be noted that the wavelengths are not harmonically

related in general, and the disturbance behind P is therefore not periodic in ξ.

F. The waves behind the surface pressure carry off energy, giving rise to a "resistance", R, which must be overcome in moving the pressure distribution. In the case of constant density the resistance is proportional to the height of the (single) wave train far behind the disturbance. In the present case the internal waves also contribute to R, and their effect can be sizable as noted in the Introduction.

The resistance can be found by summing the horizontal components of the pressure, P, over the surface deformation, η, created by P. In the linearized approximation this gives

$$(70) \qquad R = -\int_{-\ell}^{\ell} P(\xi)\eta'(\xi)d\xi$$

and in view of

$$(71) \qquad \eta(\xi) = -\lim_{y=0} U^{-1}\psi(\xi,y) = \lim_{y=0} \frac{1}{\rho_o U^2}\int_{-\ell}^{\ell} P(s)K(\xi,s;y)ds$$

we get

$$(72) \qquad R = -\frac{1}{\rho_o U^2}\lim_{y=0}\int_{-\ell}^{\ell} P(\xi)d\xi\int_{-\ell}^{\ell} P(s)K_\xi(\xi,s;y)ds$$

We cannot set y = 0 under the integral sign because of the singularity in $K_\xi(\xi,s;y)$, which behaves like a double layer charge distribution. However, the limit can be easily obtained. Let K_ξ be split up into an even and an odd function:

$$(73) \qquad K_\xi(\xi,s;y) = E(\xi,s;y) + O(\xi,s;y)$$

where $E(\xi,s;y) = \frac{1}{2}[K_\xi(\xi,s;y) + K_\xi(-\xi,-s;y)]$

and $O(\xi,s;y) = \frac{1}{2}[K_\xi(\xi,s;y) - K_\xi(-\xi,-s;y)]$

Clearly, for every $y < 0$ the odd part of K_ξ does not contribute to the integral; thus

$$(74) \qquad R = -\frac{1}{\rho_0 U^2} \lim_{y=0} \int_{-\ell}^{\ell} \int_{-\ell}^{\ell} P(\xi)\, P(s)\, E(\xi,s;y)\, d\xi ds$$

making use of (53), we obtain

$$(75) \qquad E(\xi,s;y) = \frac{1}{2\pi} \int_L \frac{k\, Y(y;k)}{[Y'-\mu Y]_0} \cos k(\xi-s)\, dk$$

Since Y and Y' are even functions of K, the integrand is odd, and the integral over the part of L which coincides with the real axis vanishes. What remains is the integral over the semicircles, which can be evaluated since we know the residues at $k = \pm k_n$. Carrying out this computation results in

$$(76) \qquad E(\xi,s;y) = -\frac{1}{2}\rho_0 \sum_{k_n>0} \frac{\phi_n(0)\, \phi_n(y)}{\|\phi_n\|_1^2} \cos k_n(\xi-s)$$

and

$$(77) \qquad R = \frac{1}{2U^2} \lim_{y=0} \int_{-\ell}^{\ell} \int_{-\ell}^{\ell} P(\xi) P(s) \sum_{k_n > 0} \frac{\phi_n(0)\phi_n(y)}{\|\phi_n\|_1^2} \cos k_n(\xi-s)$$

$$= \frac{1}{2U^2} \sum_{k_n > 0} \int_{-\ell}^{\ell} \int_{-\ell}^{\ell} P(\xi)\, P(s) [\operatorname{Re} e^{ik(\xi-s)}]\, \frac{\phi_n^2(0)}{\|\phi_n\|_1^2}$$

$$= \frac{1}{2U^2} \sum_{k_n > 0} \frac{\phi_n^2(0)}{\|\phi_n\|_1^2} |F(k_n)|^2$$

where, as before, F(k) is the Fourier Transform of $P(\xi)$ and the summation is over the positive k_n.

G. It is natural to ask whether R can vanish. For the case of constant density one can argue this possibility very simply. Take any two identical pressure distributions one half of a wavelength apart. The trailing waves produced by each one will differ in phase by 180° and will cancel each other far downstream. Consequently, the waves will vanish both upstream and downstream, and the resistance must vanish. When there are two or more waves, and the resulting wave train is not periodic, this argument fails. However, (77) shows clearly that $R = 0$, if and only if, $F(k_n) = 0$ for $n = 0, 1, \ldots, N$. This in effect constitutes $2(N+1)$ orthogonality conditions.

$$\int_{-\ell}^{\ell} P(\xi)\left[\begin{matrix}\cos\\ \sin\end{matrix}\right] k_n\xi . \, d\xi = 0, \qquad n = 0, 1, \ldots, N.$$

There is an infinite number of independent differentiable functions in the space orthogonal to $\cos k_n\xi$ and $\sin k_n\xi$ $(n = 0, 1, \ldots, N)$ over $(-\ell, \ell)$. A linear combination of any three of them can be made to vanish at $\xi = \pm \ell$. Consequently there exist pressure distributions of the type we have admitted which produce no resistance (at one speed).

Remark: The resistance will never vanish (below the highest critical speed) if $F(k) > 0$ for all k. One can construct examples for which this is true. Take any continuous and differentiable $P_\ell(\xi)$, which vanishes outside of $|\xi| < \frac{1}{2}\ell$. Form the convalution $P_\ell * P_\ell = \int_{-\ell}^{\ell} P(s)\, P(\xi - s)\, ds$ and then square it. The function $(P_\ell * P_\ell)^2$ is an admissible pressure, and, as can be easily computed, its Fourier Transform is positive for all k.(*)

(*) This example was suggested by Dr. R. Silverman.

Section III. Travelling surface pressure; time dependent solution.

A. Suppose the fluid is at rest and that a travelling surface pressure, $P = P(x+Ut)$, is applied at $t = 0$. We want to determine the time dependent stream function, i.e. determine $\psi(x,y,t)$ satisfying the differential equation (9) and the boundary condition (10), (11), the condition that ψ be bounded at ∞, and the initial conditions

$$\psi(x,y,0) = \psi_t(x,y,0) = 0 \,. \tag{78}$$

When P is not sufficiently differentiable, (1) is replaced by another condition, which need not be specified for the present problem). It is clear on physical grounds that both of the conditions (78) are required, since the initial density distribution must be known as well as the initial velocity field.

Since the steady state stream function, $\Psi(x+Ut,y)$, given by (46) or (50), satisfies the differential equation and all the boundary conditions, it is necessary only to find a solution ψ' to the problem with the homogeneous boundary condition

$$\psi'_{ytt} - g\psi'_{xx} = 0 \quad \text{at } y = 0 \tag{79}$$

and the initial conditions

$$\psi'(x,y,0) + \Psi(x,y) = 0 \tag{80}$$

$$\psi'_t(x,y,0) + U\,\Psi_x(x,y) = 0 \,.$$

Then,

$$\psi(x,y,t) = \psi'(x,y,t) + \Psi(x+Ut,y) \,.$$

The function ψ' can be found by superposing known solutions with $\nu_n^2 = g/\mu_n$, we know that $e^{ikx}e^{\pm ik\nu_n t}Y_n(y;k)$ is a solution of the

differential equation with the homogeneous boundary conditions provided $Y_n(y;k)$ satisfies

(81) $$(\rho Y_n')' - k^2\rho Y_n = \mu_n \rho' Y_n = \frac{g}{\nu_n^2} \rho' Y$$

(82) $$Y_n' - \frac{g}{\nu_n^2} Y_n = 0 \qquad \text{at } y = 0$$

(83) $$Y_n = 0 \qquad \text{at } y = -h .$$

Now, it can be easily verified that a solution to the problem is given (at least formally) by

(84) $$\psi = -(2\pi\rho_o U)^{-1} \int_L \frac{F(k)Y(y;k)}{[Y' - \mu Y]_o} e^{ik(x+Ut)} dk$$

$$+(2\pi\rho_o U)^{-1} \int_L \frac{F(k)e^{ikx}}{[Y' - \mu Y]_o} \sum a_n(k)Y_n(y;k)\zeta_n(k,t)dk$$

with

$$\zeta_n(k,t) = \frac{1}{2\nu_n}\left\{(\nu_n + U)e^{ik\nu_n t} + (\nu_n - U)e^{-ik\nu_n t}\right\}.$$

Since $\zeta_n(k,0) = 1$ and $\dfrac{\partial\zeta_n(k,0)}{\partial t} = ikU$, the initial conditions will be satisfied provided

(85) $$Y(y;k) = \sum_0^\infty a_n(k)Y_n(y;k)$$

where the Fourier coefficients, $a_n(k)$, are obtained from (23).

In view of the definition of $Y(y;k)$ as the solution to (44) taking on the initial values (45), we can evaluate the $a_n(k)$ by using Green's Formula:

$$\int_{-h}^{0} (Y_n L_2 Y - Y L_2 Y_n)dy = [\rho(YY_n' - Y'Y_n)] \Big|_{-h}^{0}$$

$$= -(\mu - \mu_n) \int_{-h}^{0} \rho' Y Y_n dy .$$

Therefore

$$\int_{-h}^{0} \rho' Y Y_n dy + \rho_o Y(0)Y_n(0) = \rho_o Y_n(0) \frac{[Y' - \mu Y]_o}{\mu_n - \mu}$$

and

$$(86) \qquad a_n(k) = (\mu_n - \mu)^{-1} \frac{\rho_o Y_n(0)[Y' - \mu Y]_o}{-\int_{-h}^{0} \rho Y_n^2 dy + \rho_o Y_n^2(0)} = (\mu_n - \mu)^{-1} \frac{Y_n(0)[Y' - \mu Y]_o}{||Y_n||_2^2}$$

Thus we finally obtain

$$(87) \qquad \psi(x,y;t) = \Psi(x + Ut, y) + \psi'(x,y;t)$$

$$(87') \qquad \Psi(x + Ut, y) = -(2\pi\rho_o U)^{-1} \int_L \frac{F(k)Y(y;k)e^{ik(x+Ut)}}{[Y' - \mu Y]_o} dk$$

$$(87'') \qquad \psi'(x,y;t) = (2\pi U)^{-1} \int_L F(k)e^{ikx} \sum_{o}^{\infty} \frac{Y_n(0;k)Y_n(y;k)}{(\mu_n - \mu)||Y_n||_2^2} \zeta_n(k,t) dk$$

This is, of course, only a formal solution.

It is interesting, at this point, to compare (87") with the analogous result for the constant density case. When ρ = const. ψ' satisfies the potential equation and only the first term in the sum in (87") remains; μ_o is determined from $\mu \tanh kh = k$ and $Y_o = \sinh k(y + h)$. The resulting integrand contains an exponential factor, $e^{|k|y}$, which permits one to differentiate the integral any number of times, and this makes it easy to prove that it represents a "genuine" solution to the problem. For $\rho \neq$ const the situation is more complicated. It is difficult to estimate the asymptotic behaviour of Y_n (i.e. for large n and all k), and the Y_n are not exponentially damped in any case ($n \geq 1$). This brings up the usual difficulty in the method of separation of variables for time dependent problems. One can prove that the right hand side of (87") is differentiable a sufficient number of times by requiring that $F(k)$ should go to zero sufficiently fast as $|k| \longrightarrow \infty$, which imposes smoothness conditions on $P(\xi)$ and its derivatives in excess of what is reasonable for the hydrodynamical problem.

We will consider only the special case where $\rho = \rho_o e^{-\beta y}$ (ρ_o = const). This reduces (81) to a differential equation with constant coefficients and makes it easy to obtain the asymptotic behaviour of Y_n. In the next section (B) it will be shown that the right hand side of (87") represents a continuous function, differentiable with respect to t and satisfying the initial conditions (80), and in section C that $\psi' \longrightarrow 0$ for every fixed y and x + Ut as $t \longrightarrow \infty$.

B. If $\rho = \rho_o e^{-\beta y}$, (81) reduces to

(88) $$Y'' - \beta Y' - k^2 Y + \beta\mu Y = 0$$

The function $e^{\frac{1}{2}\beta y} \sinh \beta(y + h)$, with $\alpha = (\frac{1}{4}\beta^2 + k^2 - \beta\mu)^{1/2}$, satisfies (88) and the condition $Y(-h;k) = 0$. The free surface condition then yields

(89) $$\frac{\tanh \alpha h}{\alpha h} = \frac{1}{(\mu - \frac{1}{2}\beta)h}$$

which determines μ_n as a function of k^2. We will need the behaviour of μ_n and Y_n when μ_n is large, i.e. for large k^2 or for large n. From (89) it is easy to obtain the following approximations

(90) $$\mu_o = |k| + 0(e^{-2|kh|})$$

$$Y_o = e^{-\frac{1}{2}\beta h} \sinh |k|(y + h) + 0(e^{-|kh|})$$

for large $|k|$

(91) $$\mu_n = \beta^{-1}[k^2 + (\tfrac{n\pi}{h})^2] + 0(1)$$

$$Y_n = e^{\frac{1}{2}\beta y} \sinh \alpha(1 + y/h) + 0(\tfrac{1}{n})$$

for large n, uniformly in k; n = 1, 2, ... ;

Then,

(92) $$\frac{Y_o(0;k)\, Y_o(y;k)}{\|Y_o\|_2^2} \simeq 2\rho_o e^{|k|y}$$

$$\frac{Y_n(0;k)\,Y_n(y;k)}{\|Y_n\|_2^2} \simeq 2(-1)^n\rho_0\,\frac{n\pi\,\sin n\pi(1+y/h)}{(n\pi)^2+(kh)^2}$$

Let μ be fixed. Then, as was shown in section 1, there is some N such that $\mu_n - \mu > 0$ for $n \geq N$ and all real k. Consequently, (87") can be rewritten in the form

$$(93) \qquad \psi' = (2\pi U)^{-1}\left\{\sum_{0}^{N-1}\int_L (\ )dk + \int_{-\infty}^{\infty} dk \sum_{N}^{\infty} (\ )\right\}$$

Since each term of the integrand is analytic in k in the neighborhood of the real axis. From (84) it follows that for real k

$$|\rho_n(k;t)| \leq 1 + \frac{U}{\nu_n} = 1 + U(\mu_n/g)^{1/2}$$

Consequently,

$$A_n = \frac{Y_n(0;k)\,Y_n(y;k)}{\|Y_n\|_2^2}\,\frac{\rho_n(k;t)e^{ikx}}{\mu_n-\mu} = O(n\mu_n^{-3/2}) = O(n^{-2})$$

for sufficiently large n. Therefore, $\sum_{M}^{\infty} A_n$ can be made arbitrarily small by choosing M sufficiently large. Furthermore, since $F(k) = O(k^{-1-\varepsilon})$, $\int_{-\infty}^{\infty} F(k)(\sum_{M}^{\infty} A_n)dk$ can be made arbitrarily small

by taking M sufficiently large, which shows that the integration and summation in (87") can be interchanged:

$$(94) \qquad \psi' = (2\pi U)^{-1}\left\{\sum_{0}^{N-1}\int_L (\ )dk + \sum_{N}^{\infty}\int_{-\infty}^{\infty} (\ )dk\right\}$$

Since ψ' is the uniform limit of continuous functions, it is continuous in (x,y,t); this shows that the first of the initial conditions in (80) is satisfied.

To prove that ψ' can be differentiated with respect to t, we observe that

$$\left|\frac{\partial \rho_n(k;t)}{\partial t}\right| \leq k\nu_n\left(1 + \frac{U}{\nu_n}\right) \quad .$$

Since $k\nu_n$ is uniformly bounded for all $n \geq 1$ (see (91)), each term of the series (after the first one) is of the same order as above and the series converges as well as the one for ψ'. Consequently ψ'_t is continuous in (x,y,t) and the second initial condition in (80) is also satisfied. It might be remarked that the estimate in (91) is also sufficient to prove that ψ' has continuous first partial derivatives with respect to x and y, without requiring any further conditions on $F(k)$.

C. We now turn to the stability question. Since the series in (94) converges uniformly with respect to x,y and t, it is sufficient to prove that each term of the series $\longrightarrow 0$ for fixed $y < 0$ and $x + Ut$ as $t \longrightarrow \infty$. Again, let N be the lowest integer for which $\mu_n - \mu > 0$ for $n \geq N$. Then we can write (87") in the form

$$(95) \qquad \psi' = \left\{ \sum_{0}^{N-1} \int_{L} dk + \sum_{N}^{\infty} \int_{-\infty}^{\infty} dk \right\} \left\{ A_n(k + Ut, y; k) e^{ik(\nu_n - U)t} + B_n(x + Ut, y; k) e^{-ik(\nu_n + U)t} \right\}$$

with

$$A_n(x + Ut, y; k) = (2\pi U)^{-1} F(k) e^{ik(x+Ut)} \frac{Y_n(0;k) Y_n(y;k)}{\|Y_n\|_2^2} \frac{\nu_n + U}{2\nu_n(\mu_n - \mu)}$$

$$B_n(x + Ut, y; k) = (2\pi U)^{-1} F(k) e^{ik(x+Ut)} \frac{Y_n(0;k) Y_n(y;k)}{\|Y_n\|_2^2} \frac{\nu_n - U}{2\nu_n(\mu_n - \mu)} \quad .$$

Consider first the terms for $n \geq N$. It has been shown in Section I that $0 < (k\nu_n)' < U$. Therefore, neither $(k\nu_n)' + U$ nor $(k\nu_n') - U$ can vanish along the real k-axis. This shows that $k(\nu_n \pm U)$ cannot have stationary points on the path of integration. Then, since $A_n = O(k^{-4-\varepsilon})$, each term can be integrated by parts:

$$\int_{-\infty}^{\infty} A_n e^{ik(\nu_n - U)t} dk = i \int_{-\infty}^{\infty} \frac{A_n' \, e^{ik(\nu_n - U)t}}{[(k\nu_n)' - U]t} \, dk$$

and it follows that the integral is $O(t^{-1})$. The same is true for the term containing B_n and for all $n \geq N$.

Now we will investigate the terms for $0 \leq n \leq N - 1$. Each one can be written in the form

$$(96) \qquad \int_L C_n \left\{ \frac{\nu_n + U}{\mu_n - \mu} e^{ik(\nu_n - U)t} + \frac{\nu_n - U}{\mu_n - \mu} e^{-ik(\nu_n + U)t} \right\} dk$$

with

$$C_n(x + Ut, y; k) = \frac{F(k) e^{ik(x + Ut)} \, Y_n(0;k) \, Y_n(y;k)}{4\pi \nu_n U \, \|Y_n\|_2^2}$$

Since

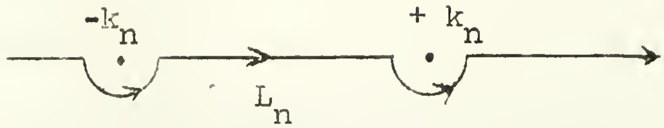
$$\frac{\nu_n \pm U}{\mu_n - \mu} = \frac{U^2 \nu_n^2 g^{-1}}{U \pm \nu_n}$$

$\dfrac{\nu_n - U}{\mu_n - \mu}$ has no poles on the real k-axis, and $\dfrac{\nu_n + U}{\mu_n - \mu}$ has poles at $k = \pm k_n$, where $\nu_n - U = 0$. For the second term in (96) we can, therefore, deform the path L into the real axis, and the resulting integral is again $O(t^{-1})$ since $k(\nu_n + U)$ has no stationary points and we can integrate by parts.

To estimate the first term in (96) we replace the path L by L_n, where L_n coincides with the real axis except in the neighborhood of $k = \pm k_n$, where it consists of semicircles of radius δ in the lower half plane. The "phase", $k(\nu_n - U)$, is stationary if

$(k\nu_n)' - U = k\nu'_n + (\nu_n - U) = 0$

As $k \to 0$, $(k\nu_n)' \to \nu_n(0)$ and $\nu_n(0) > U$. Furthermore,



as $|k| \to \infty$, $(k\nu_n)' \to 0$. Consequently there will be at least two stationary points, $k = \pm k_s$. They cannot coincide with $\pm k_n$, since this would imply $\nu_n'(\pm k_n) = 0$ which is impossible. They cannot lie outside the interval $(-k_n, + k_n)$ since $\nu_n - U < 0$ in this region and $k\nu_n' < 0$ for all real k. Therefore, they will be contained in $(-k_n + \delta, k_n - \delta)$ for sufficiently small $\delta > 0$. The contribution to the integral from this interval is $O(t^{-1/m})$ for some integer $m \geq 2$. The contribution from the intervals with $|k| > k_n + \delta$ is $O(t^{-1})$ as can be seen upon integrating by parts, and this leaves only the contributions along the semi-circles to be accounted for.

In the neighborhood of $k = \pm k_n > 0$,

$$\nu_n(k) = U + (k - k_n)\nu_n'(k_n) + \cdots$$

For $k > 0$ we know that $\nu_n' < 0$; let $\nu_n'(k_n) = -\sigma, \sigma > 0$. Along the semicircle, $k - k_n = \delta(\cos\theta + i \sin\theta)$, $\pi < \theta < 2\pi$. Therefore,

$$e^{ik(\nu_n - U)t} = e^{k_n \delta\, t(\sin\theta - i\cos\theta) + O(\delta^2 t)}$$

and since $\sin\theta < 0$ on the semicircle, the real part of the exponent is < 0, and the integral over the semicircle vanishes as $t \to \infty$. The same is true for the semicircle near $k = -k_n$. This completes the proof that the solution ψ, given by (87), tends to a steady state when observed from a coordinate system moving with the pressure disturbance. It might be noted that the estimates for each term in (87") hold for general ρ. The special form for ρ was used only to show that it suffices to consider each

Bibliography

1. Courant, R. and Hilbert, D., Methods of Mathematical Physics, Vol. 1, Interscience Publishers, New York, 1953.

2. Lamb, H., Hydrodynamics, 6th ed., Dover Publications, New York, 1945.

3. Long, R. R., Some Aspects of the Flow of Stratified Fluids, III Continuous Density Gradients, Tellus, Vol. 7, 1955, pp. 341 - 357.

4. Love, A., Wave Motion in a Heterogeneous Heavy Liquid, Proc. London Math. Society, Vol. 22, 1891, p. 307.

5. Mihlin, S. G., Variational Methods in Mathematical Physics (Russian), Moscow, 1957.

6. Peters, A. S. and Stoker, J. J., Solitary Waves in Liquids Having Non-Constant Density, IMM-NYU 259, June 1959.

7. Stoker, J. J., Water Waves Interscience Publishers, New York,1957

8. Sverdrup, H. U., and Johnson, M. W., and Fleming, R. H. The Oceans, Prentice Hall, New York, 1946.

9. Weinstein, A., On Surface Waves, Canadian Journal of Mathmatics, Vol. 1, 1949. pp. 271-278.

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